MOG → Spherical

R.F.

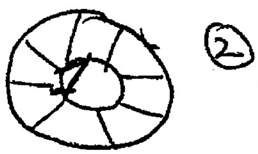
Supplement

f.) Spherical Geometry

a.) Motivation :

→ $R = R(t)$, etc. ⟺ effects of imploding system, convergence etc.

→ Multiple Surfaces :



3 stages of R.T. instability :

a.) acceleration phase : — ablated material accelerated into shell ⇒ R.T. instability at ②

$R = 3mm \rightarrow 1.5mm$

b.) coasting phase : — shell coasts inward with no acceleration → ballistic ⟺ no acceleration ⇒ no R.T.

$R = 1.5mm \rightarrow 150\mu$

c.) decelleration phase : — shell decellerates, compressing gaseous core (gas ignitor) ⇒ R.T. instability at ① (g_{eff} outward)

$R = 150\mu \rightarrow 75\mu$

1 → inner
2 → outer

Impact on ICF design:

- naively, coasting phase appears benign, as no R.T. instability

- <u>but</u> a.) during acceleration, early coasting shell radius thin

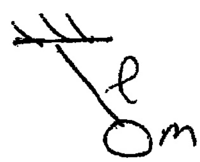
  b.) $$\varphi\Big|_{R_1} = \exp\left[-\frac{\ell}{R_1}(R_2 - R_1)\right]$$

  ⇒ outer surface R.T. seeds inner surface perturbation

  ⇒ inner surface perturbation grows due convergence <u>prior</u> to deceleration

- examples:

a.) Pendulum



with $\ell = \ell(t)$

$\dot{\ell}/\ell \ll \omega = \sqrt{g/\ell}$

Adiabatic invariant ⟷ action

$$S = \int P\,dq = \int P_\theta\,d\theta$$; here

$$H = \frac{p_\theta^2}{2mL^2} + \frac{1}{2} mgL\theta^2 \quad \Rightarrow \quad p_\theta(\theta),\ \mathcal{S} \text{ etc.}$$

Alternatively: $\mathcal{S} = $ Action
$\sim$ Energy $\times$ time

$$\mathcal{S} = \left(\tfrac{1}{2} mgLA^2\right)\left(\omega^{-1}\right)$$
$$= \frac{1}{2} m \frac{g}{L} L^2 A^2 \omega^{-1}$$
$$= \frac{1}{2} m \omega L^2 A^2$$

$\therefore \quad \mathcal{S} \sim m L^{3/2} \sqrt{g}\, A^2$

$$\Rightarrow \quad L_0^{3/2} A_0^2 = L(t)^{3/2} A(t)^2$$
$$A(t)/A_0 = \left(L_0/L(t)\right)^{3/4}$$

i.e. shortening string increases amplitude!

b.) Ocean Wave Impinging on Beach

Breaking!

$d = d(x)$

Now, $\omega = \omega(k, g, d(x))$ for finite depth 'y analogy with pendulum:

Action Density (Wave) $N = \mathcal{E}/\omega$ $\mathcal{E}$ = wave energy density

[aside: QM: $E = (N + 1/2)\hbar\omega$

$N \to$ # quanta

Semiclassical: $E = N\hbar\omega$

Classical: $\hbar \to 1$, $N \to$ action $\Rightarrow N = \mathcal{E}/\omega$]

Then: - action density (= # waves) conserved along wave trajectories

- $\frac{\partial}{\partial t} N + \underline{\nabla} \cdot (\underline{V}_{gr} N) = 0$

$$\rightarrow V_{gr}(d(x)) \frac{\mathcal{E}(x)}{\omega(d(x))} = \text{const}$$

- Key Point:

- during coasting phase, inner surface is RT stable but

- supports surface waves, seeded by outer surface RT perturbations

- as surface wave ~ harmonic oscillator can expect growth as R shrinks during coasting

i.e.

Pendulum	Inner Surface Wave
$\omega = \sqrt{g/L}$	$\omega = \left(-\frac{\ell}{R_1}\ddot{R}_1\right)^{1/2}$
m	$M = \rho V$
	$= \rho\, 4\pi R_1^2 \Delta R_{pert.}$
	$\Delta R_{pert} = R_1/\ell = k^{-1}$
	$M = \rho\, 4\pi R_1^3/\ell$

$\therefore$ $S = \frac{1}{2} m \omega L^2 A^2$

$$\rightarrow \frac{1}{2}\, 4\pi\rho \frac{R_1^3}{\ell}\left(-\frac{\ell}{R_1}\ddot{R}_1\right)^{1/2}\eta^2$$

$$\simeq \rho \frac{R_1^3}{\ell}\omega\,\eta^2$$

$$\eta^2 \sim \frac{S_o l}{\rho R_1^3 \omega} \sim \frac{\text{Const.}}{R_1^{5/2}(-\ddot{R}_1)^{1/2}}$$

$$\boxed{\eta \sim (-\ddot{R}_1)^{-1/4} R_1^{-5/4}}$$

$\Rightarrow$ perturbation grows by $(\times 10)$ during coasting phase!

References: K. O. Mikaelian; Phys. Rev. A 42 3400
M. S. Plesset; J. Appl. Phys. 25 96 (1954)
* D. L. Book, S. E. Bodner; Phys. Fl. 30 367 (19

b.) Analysis

i.) Coasting Shell "Equilibrium"

— Mass conserved during implosion

$$M = 4\pi \int_{R_1}^{R_2} \bar{\rho}(R) R^2 dr \qquad \bar{\rho} \equiv \text{avg. density}$$

$$= \frac{4\pi}{3} \bar{\rho} (R_2^3 - R_1^3)$$

$$= (4\pi/3)\, \bar{\rho}\, R_0^3 \qquad (R_0 \equiv \text{radius } \underline{\text{fully}} \text{ collapsed shell})$$

— shell incompressible (?!) — volume conserved

$$R^2 V = R^2 \dot{R} = R_1^2 \dot{R}_1$$

$$\dot{R} = \frac{R_1^2 \dot{R}_1}{R^2}$$

— For total energy: (K.E. of implosion)

$$W = \frac{1}{2}\, 4\pi \int_{R_1}^{R_2} \rho(R)\, R^2 dR\, \dot{R}^2$$

$$= 2\pi \int_{R_1}^{R_2} \bar{\rho}\, R^2 \frac{R_1^4}{R^4} \dot{R}_1^2 dR$$

$$W = 2\pi \bar{\rho}\, \dot{R}_1^2 R_1^3 \left(1 - R_1/R_2\right)$$

— time scale:

$$T_{implosion} \approx \left(M R_0^2 / 2W\right)^{1/2}$$ { mass, radius, energy

— For $\ddot{R}_1$ (i.e. ~~dynamics~~ of implosion)

$$\dot{W} = 0 = 2\pi\bar{\rho}\left[2 \dot{R}_1 \ddot{R}_1 R_1^3 \left(1 - R_1/R_2\right)\right.$$

$$+ \dot{R}_1^2 (3\dot{R}_1)(R_1^2)(1 - R_1/R_2)$$
$$+ \dot{R}_1^2 R_1^3 \left(-\frac{\dot{R}_1}{R_2} + \frac{R_1 \dot{R}_2}{R_2^2} \right) \Bigg]$$

with $R_2 \dot{R}_2 = R_1^2 \dot{R}_1$

(crank)

$$\Rightarrow \ddot{R}_1 = \frac{-W}{4\pi\rho R_1^4} \left[3 + 2\frac{R_1}{R_2} + \left(\frac{R_1}{R_2}\right)^2 \right]$$

observe for $R_1 \ll R_2$ (thick shell limit)

$$\ddot{R}_1 \sim -\frac{c}{R_1^4}$$

$$\dot{R}_1^2 \sim \frac{c}{R_1^3} \Rightarrow dR\, R_1^{3/2} \simeq dt$$
$$R_1^{5/2} \sim (t_0 - t)$$
$$\Rightarrow R_1 \sim (t_0 - t)^{2/5}$$

implosion radius evolution.

ii.) Perturbations (RT/SW)

Take: - irrotational flow
- incompressible hydro. $\left(\ell c_s / R_1 \gg \left(-\frac{\ddot{R}_1}{R} \ell \right)^{1/2} \right)$

Then, as usual, have:

i.) $\nabla^2 \phi = 0$; $\underline{V} = \nabla \phi$

ii.) At interface $\Rightarrow$ Bernoulli Eqn.:

$$\rho \left(\frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} \right) + P_1 = 0$$

here, unperturbed V from imploding shell

$\Rightarrow$

$$\frac{\partial \tilde{\phi}}{\partial t} + \dot{R} \tilde{V}_r + \tilde{P} = 0 \qquad (1)$$

iii.) Boundary conditions at interfaces:

$$\frac{d\tilde{\eta}_j}{dt} = \tilde{V}_{r_j} + \left(\frac{\partial V}{\partial R} \right)_j \tilde{\eta}_j$$

shell expanding, here

$$\tilde{P}_j = - \left(\frac{\partial P}{\partial R} \right)_j \tilde{\eta}_j$$

shell expanding

but observe:

$$\rho \ddot{R} = -\frac{\partial P}{\partial R}$$

$$R^2 \dot{R} = R_j^2 \dot{R}_j = R^2 V$$

$$\therefore \left(\frac{\partial V}{\partial R}\right)_j = \left[\frac{R_j^2 \dot{R}_j}{(R+\Delta R)^2} - \frac{R_j^2 \dot{R}_j}{R^2}\right] / \Delta R$$

$$= -2 \frac{\dot{R}_j}{R_j}$$

$$\Rightarrow \boxed{\frac{d\tilde{\eta}_j}{dt} = \tilde{V}_{r_j} - 2\frac{\dot{R}_j}{R_j}\tilde{\eta}_j} \quad (2)$$

similarly:

$$\boxed{\tilde{P}_j = \rho \ddot{R}_j \tilde{\eta}_j} \quad (3)$$

Now, to crank:

$$\nabla^2 \phi = 0 \Rightarrow$$

$$\phi(R,t) = \sum_{l,m}\left[R_1 V_1^{l,m}(t)\left(\frac{R_1}{R}\right)^{l+1} + R_2 V_2^{l,m}(t)\left(\frac{R}{R_2}\right)^{l}\right] * Y_{l,m}(\theta,\phi)$$

Then $\underline{V} = \underline{\nabla}\phi$

$$\underline{V}(R,t) = \sum_{\ell,m}\Big[V_1 (R_1/R)^{\ell+2}\left[-\hat{r}(\ell+1) Y_{\ell,m} + R\nabla Y_{\ell,m}\right] + V_2 (R/R_2)^{\ell-1}\left(\hat{r}\,\ell\, Y_{\ell,m} + R\,\nabla Y_{\ell,m}\right)\Big]$$

Substituting ϕ, $\nabla\phi$ into Bernoulli Eqn (ℓ^{th} mode)

$$-\frac{P}{\rho} = Y_{\ell,m}\Big[\Big\{R_1\dot{V}_1 + (\ell+2)\dot{R}V_1 - (\ell+1)\dot{R}_1V_1\left(\frac{R_1}{R}\right)\Big\} * \left(\frac{R_1}{R}\right)^{\ell+1} + \Big\{R_2\dot{V}_2 - (\ell-1)\dot{R}_2V_2 + \ell\dot{R}_2V_2\left(R_2/R\right)^3\Big\}\left(\frac{R}{R_2}\right)^{\ell}\Big]$$

Furthur, taking $\tilde{V}_r$ into Eqn. (2) to relate η, V_1, V_2 :

$$\dot{\eta}_1 + 2(\dot{R}_1/R)\eta_1 = -(\ell+1)V_1 + \ell A^{\ell-1}V_2$$

$$\dot{\eta}_2 + 2(\dot{R}_2/R_2)\eta_2 = -(\ell+1)A^{\ell+2}V_1 + \ell V_2$$

$A = R_1/R_2$

Similarly, plugging ③ into Bernoulli Eqn:

$$\left(R_1 \dot{V} + \left[R_2 \dot{V}_2 + (\ell A^{-3} - \ell + 1)\dot{R}_2 V_2\right]\right) A^{\ell} = -\ddot{R}_1 \eta_1$$

$$\left(R_1 \dot{V}_1 + \left[\ell + 2 - (\ell+1) A^3\right]\dot{R}_1 V_1\right) A^{\ell+1} + (R_2 \dot{V}_2) = -\ddot{R}_2 \eta_2$$

Now:

→ during coasting phase, consider <u>thick</u> shell

$R_1 / R_2 = A \ll 1 \Rightarrow \ell,\ A \to \infty$

→ $\dot{\eta}_1 + 2(\dot{R}_1/R_1)\eta_1 = -(\ell+1) V_1$

$R_1 \dot{V}_1 = -\ddot{R}_1 \eta_1$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{1}{R_1}\frac{d}{dt}(R_1^2 \tilde{\eta}_1)\right) = (\ell+1)\ddot{R}_1 \tilde{\eta}_1}$$

Similarly

$$\ddot{\eta}_2 + 2(\dot{R}_2/R_2)\,\dot{\eta}_2 \doteq \ell V_2$$

$$\dot{V}_2 = -\frac{\ddot{R}_2}{R_2}\,\eta_2$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{1}{R_2}\frac{d}{dt}\left(R_2^2\eta_2\right)\right) = -\ell\,\ddot{R}_2\,\eta_2}$$

$\Rightarrow$ two interfaces decouple!!.

For inner interface (1):

$\rightarrow$ coasting shell $\quad R_1 \sim (t_0 - t)^{2/5}$

$$\eta_1 \sim (t_0 - t)^{\alpha} \quad \text{and}$$

$$\frac{d}{dt}\left(\frac{1}{R_1}\frac{d}{dt}\left(R_1^2\eta_1\right)\right) = (\ell+1)\,\ddot{R}_1\,\eta_1$$

$$\Rightarrow \alpha = -\frac{1}{10} \pm (25 - 24\ell)^{1/2}$$

$$\therefore \eta_1 \sim (t_0 - t)^{-1/10}$$ [slow increase due convergence]

note: consequence spherical convergence, not exponential growth.

> Wave-like solution (WKB)

$$\frac{d}{dt}\left(\frac{1}{R_1}\frac{d}{dt}\left(R_1^2\tilde{\eta}_1\right)\right) = (\ell+1)\ddot{R}\,\tilde{\eta}_1$$

$$\eta_1(t) = Z(t)\, e^{i\int^t \omega(t')\,dt'}$$

$\therefore$

$$-R_1\omega^2 Z + i\left(3\omega\,\dot{R}_1 Z + 2\omega\, R\dot{Z}_1 + \dot{\omega} R_1 Z\right)$$
$$+ R_1\ddot{Z} + 3\dot{R}\dot{Z} + 2\ddot{R}_1 Z = \;=\; (\ell+1)\ddot{R}Z$$

lowest order: $\omega^2 = -\dfrac{\ddot{R}_1}{R}(\ell+1)$

$\rightarrow$ eigen frequency

first order:

$$3\omega\,\dot{R}_1 Z + 2\omega\, R\dot{Z}_1 + \dot{\omega} R_1 Z = 0$$

$$\Rightarrow \quad R_1^3 Z^2 \omega = \text{const.}$$

Recovers adiabatic invariant !!
(SW Action, surface Ⓐ)

Implications follow !

Note:

i) Question significant to Nova Upgrade $\leftrightarrow$ advisability of long coasting phase

ii) Generally, for spherical R.T.:

$$\ddot{\eta} + 3\frac{\dot{R}}{R}\dot{\eta} - n\,A(n)\,\frac{\ddot{R}}{R}\,\eta = 0$$

$n \rightarrow \ell$

$$n\,A(n) = \left[n(n-1)\,\rho_2 - (n+1)(n+2)\,\rho_1\right] / \left[n\rho_2 + (n+1)\rho_1\right]$$

need : $\quad n\,A(n)\,\ddot{R} < 0$

$$\frac{d}{dt}\left[n\,A(n)\,R^5\ddot{R}\right] < 0$$

(Plesset)

(HW?)